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COMMENT

On the Wu–Yang potentials for the Dirac monopole

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Abstract. The Kostant–Souriau theory of prequantisation is applied to the Dirac monopole, some points in a recent account of the Wu–Yang potentials in terms of the Hopf map $S^3 \to S^2$ being thereby clarified.

In a recent paper, Ryder (1980) has given an account of the Dirac monopole based on the Hopf map $S^3 \to S^2$. His account, which is briefly repeated below, contains several points worth treating at greater length and from a somewhat different perspective.

The magnetic field $\boldsymbol{B}$ of a magnetic monopole of strength g at the origin in $\boldsymbol{R}^3$ defines a two-form on $\boldsymbol{R}^3 - \{\boldsymbol{0}\}$ (since the field is singular at the origin), which is closed (since div $\boldsymbol{B} = 0$) but not globally exact (since its integral over S^2, the unit sphere centred at the origin, is $4\pi g$ and not zero). There is thus no globally defined vector potential for $\boldsymbol{B}$. The two-form is completely determined by its restriction to S^2, where it coincides with $g\sigma_2$, the two-form σ_2 being the area two-form on S^2, likewise closed but not exact. However, there are local one-forms, defined on $S^2 - \{\text{north pole}\}$ and $S^2 - \{\text{south pole}\}$ respectively, differing by an exact form on their common domain, whose exterior derivatives are both $g\sigma_2$; these local one-forms determine the Wu–Yang vector potentials for $\boldsymbol{B}$ (Wu and Yang 1975). The Hopf map $S^3 \to S^2$ defines S^3 as a non-trivial fibre bundle over S^2, with fibre the circle S^1. The pull-back of $g\sigma_2$ to S^3 by the Hopf map, denoted by B, is exact, as is every closed two-form on S^3. There are obvious local sections of S^3 over $S^2 - \{\text{north pole}\}$ and $S^2 - \{\text{south pole}\}$ by means of which the one-form A on S^3 such that $B = \mathrm{d}A$ may be pulled back to give local one-forms on S^2; these are the local one-forms which determine the Wu–Yang potentials. The quantisation condition arises from the observation that $-(\mathrm{i}e/\hbar c)A$ may be considered as a connection form, and $-(\mathrm{i}e/\hbar c)B$ its curvature form, and applying the Gauss–Bonnet–Chern theorem. It is claimed that the condition obtained is $eg = \hbar c$, the Schwinger condition, rather than the full Dirac condition $eg = \frac{1}{2}n\hbar c$ for integral n.

The final result appears to be at fault. To see why the full Dirac condition has the topological origin that it was the purpose of Ryder's paper to reveal, it is advantageous to consider the Dirac monopole in the light of the Kostant–Souriau theory of quantisation (Kostant 1970, Woodhouse 1980).

The two-form on $M = \boldsymbol{R}^3 - \{\boldsymbol{0}\}$ determined by the magnetic field $\boldsymbol{B}$ of the monopole will be denoted by $\omega_{\boldsymbol{B}}$: in fact $\omega_{\boldsymbol{B}} = \rho^*(g\sigma_2)$, where $\rho : \boldsymbol{R}^3 - \{\boldsymbol{0}\} \to S^2$ is the obvious projection from $\boldsymbol{0}$. The phase space for the motion of an electrically charged particle, of charge e, in the magnetic field of the monopole is the cotangent bundle $T^*(M)$. The two-form $\Omega_{\boldsymbol{B}} = \mathrm{d}\theta + (e/c)\tau^*\omega_{\boldsymbol{B}}$ on $T^*(M)$, where θ is the canonical one-form $p_i\,\mathrm{d}x^i$ and $\tau : T^*(M) \to M$ is the projection, defines a symplectic structure on $T^*(M)$. The motion

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of the charged particle is given by Hamilton's equations determined by this symplectic structure and the Hamiltonian $H(x, p)=\frac{1}{2}|p|^2$. According to the Kostant–Souriau theory, quantisation, or strictly speaking prequantisation, of this symplectic structure requires a complex line bundle over $T^*(M)$ (whose sections are wavefunctions), a Hermitian fibre metric on the bundle, and a connection compatible with the Hermitian structure (in terms of which the operators corresponding to classical observables are defined) whose curvature, which is a pure imaginary valued two-form on $T^*(M)$, coincides with $-(\mathrm{i}/\hbar)\Omega_{\boldsymbol{B}}$. The necessary and sufficient condition for the existence of such a Hermitian complex line bundle with connection is that $\int_S \Omega_{\boldsymbol{B}}$ over every oriented closed two-surface S in $T^*(M)$ be an integer multiple of $2\pi\hbar$. One argument for this conclusion is very similar in form, though not in interpretation, to the argument in § 3 of Ryder's paper, in which Stokes' theorem is applied to the equator of S^2, the northern and southern hemispheres lying in the domains of different Wu–Yang potentials. In the case in hand, the line integral (along a closed curve in the two-surface S) gives the phase change produced by parallel translation around the closed curve; on the application of Stokes' theorem it is the curvature two-form which must be integrated over the two parts into which S is divided by the curve (see Woodhouse (1980), ch 5 for the details). Now $\Omega_{\boldsymbol{B}} = \mathrm{d}\theta + (e/c)\tau^*\omega_{\boldsymbol{B}}$, and $\int_S \mathrm{d}\theta = 0$ for any closed oriented two-surface S. Thus $\int_S \Omega_{\boldsymbol{B}} = (e/c)\int_S \tau^*\omega_{\boldsymbol{B}}$. In effect, the only situation which does not automatically lead to a value of zero for this integral occurs when the surface S projects on a surface in $R^3-\{\boldsymbol{0}\}$ surrounding the origin, which may be smoothly deformed onto S^2, and then $\int_S \tau^*\omega_{\boldsymbol{B}} = \int_{S^2} g\sigma_2$. The quantisation condition is thus that $(eg/c)\int_{S^2}\sigma_2$ be an integral multiple of $2\pi\hbar$, i.e. that $eg=\frac{1}{2}n\hbar c$, the Dirac condition.

The Dirac condition thus has a topological or geometrical origin: it is the necessary and sufficient condition for the existence of a prequantum complex line bundle with the appropriate properties; it may be expressed by saying that $(1/2\pi\hbar)\Omega_{\boldsymbol{B}}$ must define an integral class in the de Rham cohomology group $\mathrm{H}^2(T^*(M), R)$.

It happens that for each integer n there is, up to equivalence, exactly one prequantum complex line bundle whose curvature is $-(\mathrm{i}/\hbar)\Omega_{\boldsymbol{B}}$, where the strength of $\boldsymbol{B}$ is $\frac{1}{2}n\hbar c/e$. A representative bundle may be constructed by taking advantage of the Hopf map as follows. The bundle is constructed as the pull-back of a bundle over S^2 by the map $\rho\circ\tau\colon T^*(M)\to S^2$. The group SU(2) acts transitively by bilinear transformations on S^2, the isotropy group of the south pole being U(1); thus SU(2) is a principal fibre bundle over S^2 with fibre U(1). The underlying manifold of SU(2) is just S^3, and the projection $\mathrm{SU}(2)\to S^2$ giving the principal bundle structure is the Hopf map. There is a unique connection in this principal bundle which is invariant under the action of SU(2); its connection form is, up to a constant factor, the one-form A on S^3 given by Ryder. Given any unitary representation of U(1) on C, that is to say, any character of U(1), one may form a complex line bundle over S^2 by using the associated bundle construction beginning with the principal bundle $\mathrm{SU}(2)\to S^2$ and using that representation (Kobayashi and Nomizu 1963). Such a complex line bundle will have a Hermitian structure, since a unitary representation is used to construct it, and will inherit a connection from the connection on the principal bundle. The characters of U(1) are given by the integers, the nth character being

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} \mapsto \mathrm{e}^{\mathrm{i}n\phi/2}.$$

The standard sections of SU(2) over $S^2-\{\text{north pole}\}$ and $S^2-\{\text{south pole}\}$ determine local sections of the associated complex line bundles, the local sections ψ_1, ψ_2 of the complex line bundle determined by the nth character being related on their common domain by

$$\psi_2(\theta, \phi) = e^{in\phi}\psi_1(\theta, \phi).$$

It may be shown by direct computation that the curvature of the connection in this bundle is $-\frac{1}{2}in\sigma_2$. If α_1 and α_2 are the connection one-forms for this connection based on ψ_1 and ψ_2 respectively, then $(i\hbar c/e)\rho^*\alpha_1 = A_1$ and $(i\hbar c/e)\rho^*\alpha_2 = A_2$ are real one-forms on $R^3 - \{\text{non-negative } z \text{ axis}\}$ and $R^3 - \{\text{non-positive } z \text{ axis}\}$ respectively whose exterior derivatives are both ω_B; the corresponding vector fields (defined by the standard metric on R^3) are the Wu–Yang vector potentials. The prequantum complex line bundle on $T^*(M)$ is obtained by pulling back the S^2 bundle, and the required connection has connection forms $-(i/\hbar)[\theta + (e/c)\tau^*A_1]$ and $-(i/\hbar)[\theta + (e/c)\tau^*A_2]$.

It may be of interest that sections of the complex line bundle over S^2 associated with $SU(2) \to S^2$ by the nth character of U(1) are 'functions of spin-weight n' in the terminology of Newman and Penrose (1966), and that the connections in these bundles are very closely related to the operators ð and $\bar{ð}$ introduced by them.

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